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A complete solution to the chromatic equivalence class of graph $\overline{B_{n-7,1,3}}$ ☆

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Abstract

By $h(G, x)$ and $P(G, \lambda)$ we denote the adjoint polynomial and the chromatic polynomial of a graph G, respectively. A new invariant of graph G, which is the fourth character $R_4(G)$, is given. By the properties of the adjoint polynomials, the adjoint equivalence class of graph $B_{n-7,1,3}$ is determined. According to the relations between $h(G, x)$ and $P(G, \lambda)$, we also simultaneously determine the chromatic equivalence class of $\overline{B_{n-7,1,3}}$ which is the complement of $B_{n-7,1,3}$.

Keywords: Adjoint polynomial; The smallest real root; The second smallest real root; The fourth character; Chromatic equivalence class

1. Introduction

All graphs considered here are finite and simple. Notations and terminology not defined here will conform to those in [2]. For a graph G, let $V(G)$, $E(G)$, $p(G)$, $q(G)$ and $\overline{G}$, respectively, be the set of vertices, the set of edges, the order, the size and the complement of G.

For a graph G, we denote by $P(G, \lambda)$ the chromatic polynomial of G. A partition $\{A_1, A_2, \ldots, A_r\}$ of $V(G)$, where r is a positive integer, is called an *r-independent partition* of a graph G if every A_i is a nonempty independent set of G. We denote by $\alpha(G, r)$ the number of r-independent partitions of G. Thus the chromatic polynomial of G is $P(G, \lambda) = \sum_{r \geqslant 1} \alpha(G, r)(\lambda)_r$, where $(\lambda)_r = \lambda(\lambda-1)(\lambda-2)\cdots(\lambda-r+1)$ for all $r \geqslant 1$. The readers can turn to [20] for details on chromatic polynomials.

Two graphs G and H are said to be *chromatically equivalent*, denoted by $G \sim H$, if $P(G, \lambda) = P(H, \lambda)$. By $[G]$ we denote the equivalence class determined by G under "$\sim$". It is obvious that "$\sim$" is an equivalence relation on the family of all graphs. A graph G is called chromatically unique (or simply *χ-unique*) if $H \cong G$ whenever $H \sim G$. See [4,9,10] for many results on this field.

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Definition 1.1 (*Dong et al.* [4], *Liu* [17]). Let graph G with p vertices, the polynomial

$$h(G,x)=\sum_{i=1}^{p}\alpha(\overline{G},i)x^i$$

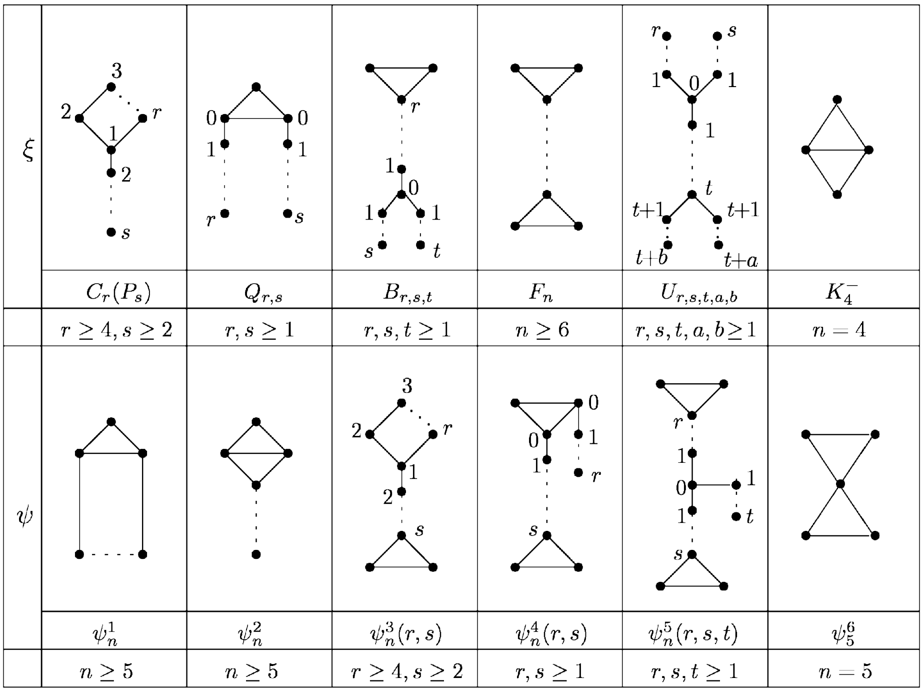
is called its adjoint polynomial.

Definition 1.2 (*Liu and Zhao* [18]). Let G be a graph and $h_1(G,x)$ the polynomial with a nonzero constant term such that $h(G,x)=x^{\rho(G)}h_1(G,x)$. If $h_1(G,x)$ is an irreducible polynomial over the rational number field, then G is called *irreducible graph.*

Two graphs G and H are said to be adjointly equivalent, denoted by $G\overset{h}{\sim}H$, if $h(G,x)=h(H,x)$. Evidently, "$\overset{h}{\sim}$" is an equivalence relation on the family of all graphs. Let $[G]_h=\{H|H\overset{h}{\sim}G\}$. A graph G is said to be adjointly unique (or simply *h-unique*) if $H\cong G$ whenever $H\overset{h}{\sim}G$.

Theorem 1.1 (*Dong et al.* [5]). (1) $G\overset{h}{\sim}H$ *if and only if* $\overline{G}\sim\overline{H}$.
(2) $[G]_h=\{H|\overline{H}\in[\overline{G}]\}$.
(3) G *is* χ-unique *if and only if* h-unique.

The graphs with orders n used in the paper are drawn as follows:



Now we define some classes of graphs, which will be used throughout the paper.

(1) C_n(resp. P_n) denotes the cycle (resp. the path) of order n, and write $\mathscr{C}=\{C_n|n\geqslant 3\}$, $\mathscr{P}=\{P_n|n\geqslant 2\}$ and $\mathscr{U}=\{U(1,1,t,1,1)|t\geqslant 1\}$.
(2) $D_n(n\geqslant 4)$ denotes the graph obtained from C_3 and P_{n-2} by identifying a vertex C_3 with a pendent vertex of P_{n-2}.
(3) T_{l_1,l_2,l_3} is a tree with a vertex v of degree 3 such that $T_{l_1,l_2,l_3}-v=P_{l_1}\cup P_{l_2}\cup P_{l_3}$ and $l_3\geqslant l_2\geqslant l_1$, write $\mathscr{T}^0=\{T_{1,1,l_3}|l_3\geqslant 1)$ and $\mathscr{T}=\{T_{l_1,l_2,l_3}|(l_1,l_2,l_3)\neq(1,1,1)\}$.
(4) $\vartheta=\{C_n,D_n,K_1,T_{l_1,l_2,l_3}|n\geqslant 4\}$.

(5) $\xi = \{C_r(P_s), Q_{r,s}, B_{r,s,t}, F_n, U_{r,s,t,a,b}, K_4^-\}$.
(6) $\psi = \{\psi_n^1, \psi_n^2, \psi_n^3(r,s), \psi_n^4(r,s), \psi_n^5(r,s,t), \psi_5^6\}$.

For convenience, we simply denote $h(G,x)$ by $h(G)$ and $h_1(G,x)$ by $h_1(G)$. By $\beta(G)$ and $\gamma(G)$ we denote the smallest and the second smallest real root of $h(G)$, respectively. Let $d_G(v)$, simply denoted by $d(v)$, be the degree of vertex v. For two graphs G and H, $G \cup H$ denotes the disjoint union of G and H, and mH stands for the disjoint union of m copies. By K_n we denote the complete graph with order n, let $n_G(K_3)$ and $n_G(K_4)$ denote the number of subgraphs isomorphic to K_3 and K_4, respectively. Let $g(x)|f(x)$ (resp. $g(x)\nmid f(x)$) denote $g(x)$ divides $f(x)$ (resp. $g(x)$ does not divide $f(x)$) and $\partial(f(x))$ denote the degree of $f(x)$. By $(f(x), g(x))$ we denote the largest common factor of $f(x)$ and $g(x)$ on the real field.

It is an interesting problem to determine $[G]$ for a given graph G. From Theorem 1.1, it is not difficult to see that the goal of determining $[G]$ can be realized by determining $[\overline{G}]_h$. The related topics have been partially discussed in this respect by Dong et al in [5]. In this paper, using the properties of adjoint polynomials, we determine the $[B_{n-7,1,3}]_h$ for graph $B_{n-7,1,3}$, simultaneously, $[\overline{B_{n-7,1,3}}]$ is also determined, where $n \geqslant 8$.

2. Preliminaries

For a polynomial $f(x) = x^n + b_1x^{n-1} + b_2x^{n-2} + \cdots + b_n$, we define

$$R_1(f(x)) = \begin{cases} -\binom{b_1}{2} + 1 & \text{if } n = 1, \\ b_2 - \binom{b_1 - 1}{2} + 1 & \text{if } n \geqslant 2. \end{cases}$$

For a graph G, we write $R_1(G)$ instead of $R_1(h(G))$.

Definition 2.1 (*Dong et al.* [3], *Liu* [17]). Let G be a graph with p vertices and q edges. The first character of G is defined as

$$R_1(G) = \begin{cases} 0 & \text{if } q = 0, \\ b_2(G) - \binom{b_1(G) - 1}{2} + 1 & \text{if } q > 0. \end{cases}$$

The second character of G is defined as

$$R_2(G) = b_3(G) - \binom{b_1(G)}{3} - (b_1(G) - 2)\left(b_2(G) - \binom{b_1(G)}{2}\right) - b_1(G),$$

where $b_i(G) = \alpha(\overline{G}, p - i)$ $(i = 1, 2, 3)$.

Lemma 2.1 (*Dong et al.* [3], *Liu* [17]). *Let G be a graph with k components $G_1, G_2, \ldots, G_k$. Then*

$$h(G) = \prod_{i=1}^{k} h(G_i) \quad \textit{and} \quad R_j(G) = \sum_{i=1}^{k} R_j(G_i) \ \textit{for}\ j = 1, 2.$$

It is obvious that $R_j(G)$ is an invariant of graphs. So, for any two graphs G and H, we have $R_j(G) = R_j(H)$ for $j = 1, 2$ if $h(G) = h(H)$ or $h_1(G) = h_1(H)$.

Lemma 2.2 (*Liu* [12,13]). *Let G be a graph with p vertices and q edges. Denote M the set of vertices of the triangles in G and by $M(i)$ the number of triangles which cover the vertex i in G. If the degree sequence of G is $(d_1, d_2, \ldots, d_p)$, then the first four coefficients of $h(G)$ are, respectively,*

(1) $b_0(G) = 1, b_1(G) = q$;
(2) $b_2(G) = \binom{q+1}{2} - \frac{1}{2}\sum_{i=1}^{p} d_i^2 + n_G(K_3)$;

(3) $b_3(G)=\frac{q}{6}(q^2+3q+4)-\frac{q+2}{2}\sum_{i=1}^{p}d_i^2+\frac{1}{3}\sum_{i=1}^{p}d_i^3+\sum_{ij\in E(G)}d_id_j-\sum_{i\in M}M(i)d_i+(q+2)n_G(K_3)+n_G(K_4)$, *where* $b_i(G)=\alpha(\overline{G},\ p-i)(i=0,1,2,3)$.

For an edge $e=v_1v_2$ of a graph G, the graph $G*e$ is defined as follows: the vertex set of $G*e$ is $(V(G)-\{v_1,v_2\})\cup\{v\}(v\notin G)$, and the edge set of $G*e$ is $\{e'|e'\in E(G), e'$ is not incident with v_1 or $v_2\}\cup\{uv|u\in N_G(v_1)\cap N_G(v_2)\}$, where $N_G(v)$ is the set of vertices of G which are adjacent to v.

Lemma 2.3 (*Liu [12]*). *Let G be a graph with $e\in E(G)$. Then*

$$h(G,x)=h(G-e,x)+h(G*e,x),$$

where $G-e$ denotes the graph obtained by deleting the edge e from G.

Lemma 2.4 (*Liu [11,15,16]*). (1) *For $n\geqslant 2$, $h(P_n)=\sum_{k\leqslant n}\binom{k}{n-k}x^k$.*

(2) *For $m\geqslant 4$, $h(D_n)=\sum_{k\leqslant n}\left(\frac{n}{k}\binom{k}{n-k}+\binom{k-2}{n-k-3}\right)x^k$.*

(3) *For $n\geqslant 4$, $m\geqslant 6$, $h(P_n)=x(h(P_{n-1})+h(P_{n-2}))$, $h(D_m)=x(h(D_{m-1})+h(D_{m-2}))$.*

Lemma 2.5 (*Zhao [23]*). *Let $\{g_i(x)\}$, simply denoted by $\{g_i\}$, be a polynomial sequence with integer coefficients and $g_n(x)=x(g_n(x)+g_{n-1}(x))$. Then*

(1) $g_n(x)=h(P_k)g_{n-k}(x)+xh(P_{k-1})g_{n-k-1}(x)$.
(2) $h_1(P_n)|g_{k(n+1)+i}(x)$ *if and only if* $h_1(P_n)|g_i(x)$, *where* $0\leqslant i\leqslant n$, $n\geqslant 2$ *and* $k\geqslant 1$.

Lemma 2.6 (*Du [6], Liu and Zhao [18]*). *Let G be a nontrivial connected graph with n vertices. Then*

(1) $R_1(G)\leqslant 1$, *and the equality holds if and only if* $G\cong P_n(n\geqslant 2)$ *or* $G\cong K_3$.
(2) $R_1(G)=0$ *if and only if* $G\in\vartheta$.
(3) $R_1(G)=-1$ *if and only if* $G\in\xi$, *especially*, $q(G)=p(G)+1$ *if and only if* $G\in\{F_n|n\geqslant 6\}\cup\{K_4^-\}$.
(4) $R_1(G)=-2$ *if and only if* $G\in\psi$ *for* $q(G)=p(G)+1$ *and* $G\cong K_4^-$ *for* $q(G)=p(G)+2$.

Lemma 2.7 (*Huo [8]*). *For $k\geqslant 0$, let $G^{(-k)}$ denote the union of the components of G whose the first characters are $-k$ and s_k denote the number of components of $G^{(-k)}$. Then*

(1) *If $k=0$, or -1, or -2, then $q(G^{(-k)})-p(G^{(-k)})\leqslant ks_k$ and the equality holds if and only if each component G_i of $G^{(-k)}$ such that $q(G_i)-p(G_i)=k$, where $1\leqslant i\leqslant s_k$.*
(2) *If $k=-3$, then $q(G^{(-k)})-p(G^{(-k)})\leqslant 2s_3$ and the equality holds if and only if each component G_i of $G^{(-3)}$ such that $q(G_i)-p(G_i)=2$, where $1\leqslant i\leqslant s_3$.*

Lemma 2.8 (*Zhao [23]*). *Let G be a connected graph and H a proper subgraph of G, then*

$$\beta(G)<\beta(H).$$

Lemma 2.9 (*Zhao [23]*). *Let G be a connected graph. Then*

(1) $\beta(G)=-4$ *if and only if*

$$G\in\{T(1,2,5),T(2,2,2),T(1,3,3),K_{1,4},C_4(P_2),Q(2,2),K_4^-,D_8\}\cup\mathscr{U}.$$

(2) $\beta(G)>-4$ *if and only if*

$$G\in\{K_1,T(1,2,i)(2\leqslant i\leqslant 4),D_i(4\leqslant i\leqslant 7)\}\cup\mathscr{P}\cup\mathscr{C}\cup\mathscr{T}^0.$$

Lemma 2.10 (*Zhao [23]*). *Let G be a connected graph. Then* $-(2+\sqrt{5})\leqslant\beta(G)<-4$ *if and only if G is one of the following graphs*:

(1) T_{l_1,l_2,l_3} *for* $l_1=1, l_2=2, l_3>5$ *or* $l_1=1, l_2>2, l_3>3$, *or* $l_1=l_2=2, l_3>2$, *or* $l_1=2, l_1=l_2=3$.
(2) $U_{r,s,t,a,b}$ *for* $r=a=1, (r,s,t)\in\{(1,1,2),(2,4,2),(2,5,3),(3,7,3),(3,8,4)\}$, *or* $r=a=1, s\geqslant 1, t\geqslant t^*(s,b)$, $b\geqslant 1$, *where* $(s,b)\neq(1,1)$ *and*

$$t^*=\begin{cases} s+b+2, & \text{if } s\geqslant 3,\\ b+3, & \text{if } s=2,\\ b, & \text{if } s=1.\end{cases}$$

(3) D_n *for* $n\geqslant 9$.
(4) $C_n(P_2)$ *for* $n\geqslant 5$.
(5) F_n *for* $n\geqslant 9$.
(6) $B_{r,s,t}$ *for* $r=5, s=1$ *and* $t=3$, *or* $r\geqslant 1, s=1$ *if* $t=1$, *or* $r\geqslant 4, s=1$ *if* $t=2$, *or* $b\geqslant c+3, s=1$ *if* $t\geqslant 3$.
(7) $G\cong C_4(P_3)$, *or* $G\cong Q_{1,2}$.

From Lemmas 2.6 and 2.10, the following corollary is obtained.

Corollary 2.1. *If graph G such that* $R_1(G)\leqslant -2$, *then* $\beta(G)<-2-\sqrt{5}$.

Lemma 2.11 (*Ren [21]*). *Let graph* $G_n\in\xi\backslash\{F_n, U_{r,s,t,a,b}, K_4^-\}$, *then*

(1) $b_3(G_n)=b_3(D_n)-n+5$ *if and only if*

$$G_n\in\{C_r(P_s)|r\geqslant 4, s\geqslant 3\}\cup\{Q_{1,n-4}|n\geqslant 6\}\cup\{B_{r,1,t}, B_{1,1,1}|r,t\geqslant 2\}.$$

(2) $b_3(G_n)=b_3(D_n)-n+6$ *if and only if*

$$G_n\in\{Q_{r,s}|r,s\geqslant 2\}\cup\{B_{1,1,t}, B_{r,s,t}|r,s,t\geqslant 2\}.$$

Lemma 2.12 (*Ren [21]*). *Let graph* $G_n\in\psi$, *then* $b_3(G_n)=b_3(D_{n+1})-2(n+1)+t$, *where* $10\leqslant t\leqslant 13$.

Lemma 2.13 (*Liu [14]*). *Let* $f(x)$ *be the monadic integral coefficients polynomial in x. If all the roots of* $f(x)$ *are nonnegative and there exists positive integer k such that* $f(k)$ *is a prime number, then* $f(x)$ *is a irreducible polynomial over the rational number field.*

3. The algebraic properties of adjoint polynomials

3.1. The divisibility of adjoint polynomials and the fourth characters of graphs

Definition 3.1.1. The adjoint roots of graph G are the roots of its adjoint polynomial $h(G)$.

Lemma 3.1.1 (*Zhao [23]*). *For* $n,m\geqslant 2$, $h(P_n)|h(P_m)$ *if and only if* $n+1|m+1$.

Theorem 3.1.1. (1) *For* $n\geqslant 8$,

$$\partial(h_1(B_{n-7,1,3}))=\begin{cases}\frac{n}{2} & \text{if } n \text{ is even},\\ \frac{n-1}{2} & \text{otherwise}.\end{cases}$$

(2) *For* $n\geqslant 8$,

$$\rho(B_{n-7,1,3})=\begin{cases}\frac{n}{2} & \text{if } n \text{ is even},\\ \frac{n+1}{2} & \text{otherwise}.\end{cases}$$

(3) *For* $n\geqslant 10$, $h(B_{n-7,1,3})=x(h(B_{n-8,1,3})+h(B_{n-9,1,3}))$.

Proof. (1) Choosing a pendant edge $e = uv \in E(B_{n-7,1,3})$ such that $d(u) = 1$, $d(v) = 3$ and by Lemma 2.3, we have $h(B_{n-7,1,3}) = xh(D_{n-1}) + xh(P_3)h(D_{n-5})$. We have, from Lemma 2.4, that

$$\partial(h_1(D_{n-1})) = \left\lfloor \frac{n}{2} \right\rfloor \quad \text{and} \quad \partial(h_1(P_3)h_1(D_{n-5})) = 1 + \left\lfloor \frac{n-4}{2} \right\rfloor.$$

If n is even, then $[\frac{n}{2}] = \frac{n}{2} > 1 + \lfloor \frac{n-4}{2} \rfloor = \frac{n-2}{2}$. If n is odd, then $\lfloor \frac{n}{2} \rfloor = \frac{n-1}{2} > 1 + \lfloor \frac{n-4}{2} \rfloor = \frac{n-3}{2}$. Hence the result holds.

(2) Obviously follows from (1).

(3) Choosing a pendant edge $e = uv \in E(B_{n-7,1,3})$ such that $d(u) = 1$, $d(v) = 3$, we have, by Lemma 2.4, that

$$\begin{aligned} h(B_{n-7,1,3}) &= xh(D_{n-1}) + xh(P_3)h(D_{n-5}) \\ &= x(xh(D_{n-2}) + xh(D_{n-3})) + xh(P_3)(xh(D_{n-6}) + xh(D_{n-7})) \\ &= x(xh(D_{n-2}) + xh(P_3)h(D_{n-6})) + x(xh(D_{n-3}) + xh(P_3)h(D_{n-7})) \\ &= x(h(B_{n-8,1,3}) + h(B_{n-9,1,3})). \qquad \square \end{aligned}$$

Theorem 3.1.2. *For $n \geqslant 2$ and $m \geqslant 8$, $h(P_n) | h(B_{m-7,1,3})$ if and only if $n = 2$ and $m = 3k + 2$ for $k \geqslant 2$ or $n = 6$ and $m = 7k + 3$ for $k \geqslant 1$.*

Proof. Let $g_0(x) = x^5 + 8x^4 + 22x^3 + 26x^2 + 13x + 3$, $g_1(x) = -x^5 - 7x^4 - 16x^3 - 15x^2 - 4x$ and $g_m(x) = x(g_{m-1}(x) + g_{m-2}(x))$. We can deduce that

$$\begin{aligned} &g_0(x) = x^5 + 8x^4 + 22x^3 + 26x^2 + 13x + 3, \\ &g_1(x) = -x^5 - 7x^4 - 16x^3 - 15x^2 - 4x, \\ &g_2(x) = x^5 + 6x^4 + 11x^3 + 9x^2 + 3x, \\ &g_3(x) = -x^5 - 5x^4 - 6x^3 - x^2, \\ &g_4(x) = x^5 + 5x^4 + 8x^3 + 3x^2, \\ &g_5(x) = 2x^4 + 2x^3, \\ &g_6(x) = x^6 + 7x^5 + 10x^4 + 3x^3, \\ &g_7(x) = x^7 + 7x^6 + 12x^5 + 5x^4, \\ &g_m(x) = h(B_{m-7,1,3}) \text{ if } m \geqslant 8. \end{aligned} \tag{3.1}$$

Let $m = (n + 1)k + i$, where $0 \leqslant i \leqslant n$. It is obvious that $h_1(P_n) | h(B_{m-7,1,3})$ if and only if $h_1(P_n) | g_m(x)$. From Lemma 2.5, it follows that $h_1(P_n) | g_m(x)$ if and only if $h_1(P_n) | g_i(x)$, where $0 \leqslant i \leqslant n$. We distinguish the following two cases:

Case 1: $n \geqslant 8$.

If $0 \leqslant i \leqslant 7$, from (3.1), it is not difficult to verify that $h_1(P_n) \nmid g_i(x)$. If $i \geqslant 8$, From $i \leqslant n$, Lemma 2.4 and Theorem 3.1.1, we have that $\partial(h_1(P_n)) = \lfloor \frac{n}{2} \rfloor \geqslant \partial(h_1(B_{i-7,1,3}) = \lfloor \frac{i}{2} \rfloor$. Suppose that $h_1(P_n) | h_1(B_{i-7,1,3})$, it must leads to $\partial(h_1(P_n)) = \partial(h_1(B_{i-7,1,3})$ and $h_1(P_n) = h_1(B_{i-7,1,3})$, which imply that $R_1(P_n) = R_1(B_{i-7,1,3})$. This is a contradiction by Lemma 2.6. Hence $h_1(P_n) \nmid h_1(B_{i-7,1,3})$, together with $(h_1(P_n), x^{\rho(B_{i-7,1,3})}) = 1$, we have that $h_1(P_n) \nmid h(B_{i-7,1,3})$.

Case 2: $2 \leqslant n \leqslant 7$.

From (1) of Lemma 2.4 and (3.1), we can verify that $h_1(P_n) | g_i(x)$ if and only if $n = 2$ and $i = 2$ or $n = 6$ and $i = 3$ for $0 \leqslant i \leqslant n \leqslant 7$. From Lemma 2.5, we have that $h_1(P_n) | h(B_{m-7,1,3})$ if and only if $n = 2$ and $m = 3k + 2$ or $n = 6$ and $m = 7k + 3$. From $\rho(P_3) = 2$, $\rho(P_6) = 3$ and $\rho(B_{m-7,1,3}) = \lfloor \frac{m}{2} \rfloor \geqslant 4$ for $m \geqslant 8$, we obtain that the result holds. $\square$

Theorem 3.1.3. *For $m \geqslant 8$, $h^2(P_2) \nmid h(B_{m-7,1,3})$ and $h^2(P_6) \nmid h(B_{m-7,1,3})$.*

Proof. Suppose that $h^2(P_2)|h(B_{m-7,1,3})$, from Theorem 3.1.2, we have that $m=3k+2$, where $k\geqslant 2$. Let $g_m(x)=h(B_{m-7,1,3})$ for $m\geqslant 8$. By (3) of Theorem 3.1.1, (1) of Lemma 2.5, it follows that

$$\begin{aligned} g_m(x) &= h(P_2)g_{m-2}(x)+x^2g_{m-3}(x)\\ &= h^2(P_2)g_{m-4}(x)+2x^2h(P_2)g_{m-5}(x)+x^4g_{m-6}(x)\\ &= h^2(P_2)(g_{m-4}(x)+2x^2g_{m-7}(x))+3x^4h(P_2)g_{m-8}(x)+x^6g_{m-9}(x)\\ &= h^2(P_2)(g_{m-4}(x)+2x^2g_{m-7}(x)+3x^4g_{m-10}(x))+4x^6h(P_2)g_{m-11}(x)+x^8g_{m-12}(x)\\ &= \cdots = h^2(P_2)\sum_{s=1}^{k-2}g_{m-3s-1}(x)+(k-1)x^{2k-4}h(P_2)g_{m+1-3(k-1)}(x)+x^{2k-2}g_{m-3(k-1)}(x). \end{aligned}$$

According to the assumption and $m=3k+2$, we arrive at, by (3.1), that

$$h^2(P_2)|((k-1)x^{2k-4}h(P_2)g_6(x)+x^{2k-1}g_5(x)),$$

that is,

$$h(P_2)|((k-1)x^{2k+2}+7(k-1)x^{2k+1}+(10k-8)x^{2k}+3(k-1)x^{2k-1}).$$

By direct calculation, we obtain that $k=-1$, which contradicts to $k\geqslant 2$.

Using the similar methods, we can also prove $h^2(P_6)\nmid h(B_{m-7,1,3})$. The details here are omitted. $\square$

Definition 3.1.2. Let G be a graph with p vertices and q edges. The fourth character of G is defined as follows:

$$R_4(G)=R_2(G)+p-q.$$

From Lemmas 2.1 and 2.2, we obtain the following two theorems:

Theorem 3.1.4. *Let graph G with k components $G_1,G_2,\ldots,G_k$. Then*

$$R_4(G)=\sum_{i=1}^{k}R_4(G_k).$$

Theorem 3.1.5. *If graphs G and H such that $h(G)=h(H)$ or $h_1(G)=h_1(H)$, then*

$$R_4(G)=R_4(H).$$

From Definitions 3.1.2 and 2.1, we have the following theorem:

Theorem 3.1.6. (1) $R_4(C_n)=0$ *for* $n\geqslant 4$ *and* $R_4(C_3)=-2$; $R_4(K_1)=1$.
(2) $R_4(B_{r,1,1})=3$ *for* $r\geqslant 1$; *and* $R_4(B_{r,1,t})=4$ *for* $r,t>1$.
(3) $R_4(F_6)=4$, $R_4(F_n)=3$ *for* $n\geqslant 7$ *and* $R_4(K_4^-)=2$.
(4) $R_4(D_4)=0$ *and* $R_4(D_n)=1$ *for* $n\geqslant 5$; $R_4(T_{1,1,1})=0$.
(5) $R_4(T_{1,1,l_3})=1$, $R_4(T_{1,l_2,l_3})=2$ *and* $R_4(T_{l_1,l_2,l_3})=3$ *for* $l_3\geqslant l_2\geqslant l_1\geqslant 2$.
(6) $R_4(C_r(P_2))=3$ *for* $m\geqslant 4$ *and* $R_4(C_4(P_3))=R_4(Q_{1,2})=4$.
(7) $R_4(P_2)=0$ *and* $R_4(P_n)=-1$ *for* $n\geqslant 3$.

3.2. The smallest real roots of adjoint polynomials of graphs

An *internal* x_1x_k*-path* of a graph G is a path $x_1x_2x_3\cdots x_k$ (possibly $x_1=x_k$) of G such that $d(x_1)$ and $d(x_k)$ are at least 3 and $d(x_2)=d(x_3)=\cdots=d(x_{x-1})$ (unless $k=2$).

Lemma 3.2.1 (*Zhao [23]*). *Let T be a tree. If uv is an edge on an internal path of T and $T\ncong U(1,1,t,1,1)$ for $t\geqslant 1$, then $\beta(T)<\beta(T_{xy})$, where T_{xy} is the graph obtained from T by inserting a new vertex on the edge xy of T.*

Lemma 3.2.2 (*Zhao [23]*). (1) *For* $n\geqslant 5$, $m\geqslant 4$, $\beta(C_n(P_2))<\beta(C_{n-1}(P_2))\leqslant\beta(D_m)$.

(2) *For* $n\geqslant 6$ *and* $m\geqslant 6$, $\beta(F_n)=\beta(B_{m-5,1,1})$ *if and only if* $n=2k+1$ *and* $m=k+2$.

(3) *For* $m\geqslant 6$ *and* $n\geqslant 4$, $\beta(F_m)<\beta(F_{m+1})<\beta(D_n)$ *and* $\beta(B_{m-5,1,1})<\beta(B_{m-4,1,1})<\beta(D_n)$.

From Lemma 2.3 and calculation, we have the following lemma:

Lemma 3.2.3. (1) $T_{1,3,6}\overset{h}{\sim}C_5(P_2)\cup P_5$, $B_{n-6,1,2}\overset{h}{\sim}K_1\cup F_{n-1}$ *for* $n\geqslant 7$.

(2) $B_{5,1,3}\cup K_{1,3}\overset{h}{\sim}K_1\cup C_{14}(P_2)$, $B_{6,1,3}\cup K_{1,3}\overset{h}{\sim}K_1\cup C_7\cup C_8(P_2)$,

(3) $B_{6,1,3}\overset{h}{\sim}C_7\cup C_4(P_3)\overset{h}{\sim}C_7\cup B_{1,1,1}\overset{h}{\sim}C_7\cup Q_{1,2}$, $B_{7,1,3}\overset{h}{\sim}D_6\cup C_7(P_2)$, $B_{10,1,3}\overset{h}{\sim}C_6(P_2)\cup P_6\cup K_4^-\overset{h}{\sim}B_{2,1,1}\cup P_6\cup K_4^-$.

(4) $h(B_{3,1,3}\cup P_4)=(x^4+5x^3+3x^2)h(P_6)h(B_{2,1,2})$, $B_{6,1,3}\cup C_4\overset{h}{\sim}B_{4,1,2}\cup C_7$, $B_{10,1,3}\cup D_5\overset{h}{\sim}C_4(P_2)\cup P_6\cup B_{6,1,2}$, $T_{1,3,11}\cup D_6\overset{\mathrm{h}}{\sim}P_3\cup C_5\cup B_{8,1,2}$.

(5) $U_{1,2,r,1,t}\overset{h}{\sim}K_1\cup B_{r,1,t}$ *for* $r,t\geqslant 1$.

Lemma 3.2.4. (1) $\beta(T_{1,3,6})=\beta(C_5(P_2))$, $\beta(1,3,11)=\beta(B_{8,1,2})$.

(2) $\beta(B_{5,1,3})=\beta(C_{14}(P_2))$, $\beta(B_{6,1,3})=\beta(C_8(P_2))$, $\beta(B_{10,1,3})=\beta(C_6(P_2))$.

(3) *For* $r,t\geqslant 1$, $\beta(B_{t,1,2})=\beta(F_{t+5})$ *and* $\beta(U(1,2,r,1,t))=\beta(B_{r,1,t})$.

(4) $\beta(B_{3,1,3})=\beta(B_{2,1,2})$, $\beta(B_{6,1,3})=\beta(B_{7,1,2})$, $\beta(B_{10,1,3})=\beta(B_{4,1,2})$.

(5) *For* $r,t\geqslant 1$, $\beta(B_{r,1,t})<\beta(B_{r+1,1,t})$.

Proof. The first four results of the theorem obviously holds from Lemma 3.2.3.

(5) By Lemma 3.2.1, we have $\beta(U_{1,2,r,1,t})<\beta(U_{1,2,r+1,1,t})$. From (3) of the lemma, we arrive at the result. □

Theorem 3.2.1. (1) *For* $n_1\geqslant 15$, $13\leqslant n_2\leqslant 9$ *and* $n\geqslant 18$, $\beta(B_{1,1,3})<\beta(B_{2,1,3})<\beta(B_{3,1,3})<\beta(C_{n_1}(P_2))<\beta(C_{14}(P_2))=\beta(B_{5,1,3})<\beta(C_{n_2}(P_2))<\beta(C_8(P_2))=\beta(B_{6,1,3})<\beta(C_7(P_2))=\beta(B_{7,1,3})<\beta(B_{8,1,3})<\beta(B_{9,1,3})<\beta(B_{10,1,3})=\beta(C_6(P_2))<\beta(B_{n-7,1,3})<\beta(C_5(P_2))<\beta(C_4(P_2))$.

(2) *For* $n\geqslant 21$ *and* $m\geqslant 14$, $\beta(B_{1,1,3})<\beta(F_6)=\beta(B_{1,1,2})<\beta(B_{2,1,3})<\beta(F_7)=\beta(B_{2,1,2})=\beta(B_{3,1,3})<\beta(B_{4,1,3})<\beta(F_8)=\beta(B_{3,1,2})<\beta(B_{5,1,3})<\beta(B_{6,1,3})=\beta(B_{4,1,2})=\beta(F_9)<\beta(B_{7,1,3})<\beta(B_{5,1,2})=\beta(F_{10})<\beta(B_{8,1,3})<\beta(B_{9,1,3})<\beta(B_{10,1,3})=\beta(B_{6,1,2})=\beta(F_{11})<\beta(B_{11,1,3})<\beta(B_{12,1,3})<\beta(B_{13,1,3})<\beta(B_{7,1,2})=\beta(F_{12})<\beta(B_{n-7,1,3})<\beta(B_{m-6,1,2})=\beta(F_{m-1})$.

(3) *For* $n\geqslant 8$ *and* $m\geqslant 6$, $\beta(B_{n-7,1,3})=\beta(B_{m-5,1,1})$ *if and only if* $n=13$, $m=6$, *or* $n=17$, $m=7$.

(4) *For* $n\geqslant 8$ *and* $m\geqslant 4$, $\beta(B_{n-7,1,3})<\beta(D_m)$.

(5) *For* $n\geqslant 8$, $\beta(B_{n-7,1,3})=\beta(Q_{1,2})=\beta(C_4(P_3))$ *if and only if* $n=13$.

(6) *For* $m\geqslant 4$, $\beta(Q_{1,2})=\beta(C_4(P_3))<\beta(D_m)$.

(7) *For* $t\geqslant 4$ and $n\geqslant m$, $\beta(B_{m-t-4,1,t})<\beta(B_{n-7,1,3})$.

Proof. (1) For $n\geqslant 18$, it is obvious that $T_{1,3,6}$ is a proper subgraph of $B_{n-7,1,3}$. From Lemma 2.8 and (1) of Lemma 3.2.4, it follows that $\beta(B_{n-7,1,3})<\beta(T_{1,3,6})=\beta(C_5(P_2))$. From (2) and (5) of Lemma 3.2.4 and (1) of Lemma 3.2.2, we have that the result holds.

(2) Using Software Mathematica and by calculation, we have that $\beta(B_{1,1,3})=-4.4605<\beta(F_6)=-4.39026<\beta(B_{2,1,3})=-4.36234$, $\beta(B_{4,1,3})=-4.26308<\beta(F_8)=-4.24978<\beta(B_{5,1,3})=-4.23499$, $\beta(B_{7,1,3})=-4.19869<\beta(F_{10})=-4.189<\beta(B_{8,1,3})=-4.18667$, $\beta(B_{13,1,3})=-4.1568<\beta(F_{12})=-4.15546<\beta(B_{14,1,3})=-4.15431$. For $n\geqslant 21$, it follows, from Lemma 2.8 and (1) of Lemma 3.2.4, that $\beta(B_{n-7,1,3})<\beta(T_{1,3,11})=\beta(B_{8,1,2})$. From (3)–(5) of Lemma 3.2.4 and (3) of Lemma 3.2.2, we have the result.

(3) From (2) of Lemma 3.2.2 and (2) of the theorem, the result obviously holds.

(4) From (2) of the theorem and (3) of Lemma 3.2.2, it is easy to get the result.

(5) From (3) of Lemma 3.2.3 and (5) of Lemma 3.2.4, the result evidently holds.

(6) It obviously follows from (4) and (5) of the theorem.

(7) Since $n\geqslant m$ and $t\geqslant 4$, from (5) of Lemma 3.2.4 and Lemma 2.8, we have that $\beta(B_{m-t-4,1,t})<\beta(B_{n-t-4,1,t})\leqslant\beta(B_{n-8,1,t})<\beta(B_{n-7,1,t})<\beta(B_{n-7,1,3})$. □

3.3. The second smallest real roots of adjoint polynomials of graphs

Lemma 3.3.1 (*Gantmacher [7]*). *Let G be a simple graph and $v \in V(G)$. If the spectrums of G and $G - v$ are, respectively, $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$* and *$\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_{n-1}$, then*

$$\lambda_1 \geqslant \mu_1 \geqslant \lambda_2 \geqslant \mu_2 \geqslant \cdots \geqslant \lambda_{n-1} \geqslant \mu_{n-1} \geqslant \lambda_n.$$

Lemma 3.3.2 (*Biggs [1]*). *Let the characteristic polynomial of a tree be $f(T, \lambda) = \sum c_i \lambda^{n-i}$, then the roots of $f(T)$ are symmetric by* 0, *where the odd coefficients c_{2r+1} are zero and the even coefficients c_{2r} are given by the rule that $(-1)^r c_{2r}$ is the number of ways of choosing r disjoint edges in the tree.*

Lemma 3.3.3 (*Zhao [22]*). *Let T be a tree or a forest. If λ and θ are, respectively, the eigenvalue and the adjoint root of T, then $\theta = -\lambda^2$.*

Theorem 3.3.1. *Let the adjoint roots of tree T and $T - v$ are, respectively, $\eta_1 \leqslant \eta_2 \leqslant \cdots \leqslant \eta_n$ and $\theta_1 \leqslant \theta_2 \leqslant \cdots \leqslant \theta_{n-1}$, where $v \in V(T)$ and $p(T) = n$.*

(1) *If $n = 2k$, then the nonzero adjoint roots of T and $T - v$ such that*

$$\eta_1 \leqslant \theta_1 \leqslant \eta_2 \leqslant \theta_2 \leqslant \cdots \leqslant \eta_{k-1} \leqslant \theta_{k-1} \leqslant \eta_k < 0.$$

(2) *If $n = 2k + 1$, then the nonzero adjoint roots of T and $T - v$ such that*

$$\eta_1 \leqslant \theta_1 \leqslant \eta_2 \leqslant \theta_2 \leqslant \cdots \leqslant \theta_{k-1} \leqslant \eta_k \leqslant \theta_k < 0.$$

Proof. (1) From $n = 2k$ and Lemma 3.3.2, if follows that $f(T)$ has no zero-root and $f(T - v)$ has only one zero-root. Without loss of generality, Let the roots of $f(T)$ and $f(T) - v$ are, respectively, $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k \geqslant -\lambda_k \geqslant -\lambda_{k-1} \geqslant \cdots \geqslant -\lambda_1$ and $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_{k-1} > 0 > -\mu_{k-1} \geqslant -\mu_{k-2} \geqslant \cdots \geqslant -\mu_1$.

According to Lemma 3.3.1, we arrive at

$$\lambda_1 \geqslant \mu_1 \geqslant \lambda_2 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_{k-1} \geqslant \lambda_k > 0 > -\lambda_k \geqslant -\mu_{k-1} \geqslant -\lambda_2 \geqslant -\mu_1 \geqslant -\lambda_1.$$

In the light of Lemma 3.3.3, we get that

$$\eta_1 \leqslant \theta_1 \leqslant \eta_2 \leqslant \theta_2 \leqslant \cdots \leqslant \eta_{k-1} \leqslant \theta_{k-1} \leqslant \eta_k < 0.$$

(2) Using the similar proof as that of (1), we can prove that the result holds. □

Theorem 3.3.2 (*Mao [19]*). (1) $\gamma(U_{1,2,t,1,3}) = -4$ *if and only if $t = 10$ for $t \geqslant 1$.*
(2) $\gamma(B_{n-7,1,3}) = -4$ *if and only if $n = 17$ for $n \geqslant 8$.*
(3) $\gamma(T_{l_1,l_2,l_3}) > -4$ *for $l_3 \geqslant l_2 \geqslant l_1 \geqslant 1$.*

Proof. (2) From (5) of Lemma 3.2.3 and (1) of the theorem, the result holds.

(3) Choosing the vertex $v \in V(T_{l_1,l_2,l_3})$ such that $d(v) = 3$ and from Lemma 2.9, we obtain, from Lemma 2.9 and Theorem 3.3.1, that $\gamma(T_{l_1,l_2,l_3}) \geqslant \beta(T - v) = \beta(P_{l_1} \cup P_{l_2} \cup P_{l_3}) > -4$. □

4. The chromaticity of the complement of graph $B_{n-7,1,3}$

Theorem 4.1. *Let graph G such that $G \overset{h}{\sim} B_{n-7,1,3}$, where $n \geqslant 8$. Then G contains at most two components whose first characters are* 1, *furthermore, one of them is P_2 and the other is P_6.*

Proof. Let G_1 be one of the components of G such that $R_1(G) = 1$. From Lemma 2.6 and $h_1(P_4) = h_1(C_3)$, it follows, from Theorem 3.1.2, that $h(G_1) | h(B_{n-7,1,3})$ if and only if $G_1 \cong P_2$ and $n = 3k + 2$, or $G_1 \cong P_6$ and $n = 7k + 3$, furthermore, $h_1(C_3) \nmid h(B_{n-7,1,3})$. From (1) of Lemma 2.5, we obtain the following equality:

$$h(B_{21k+10,1,3}) = h(P_{21})h(B_{21(k-1)+10,1,3}) + xh(P_{20})h(B_{21(k-1)+19,1,3}). \tag{4.1}$$

Noting that $\{n|n=3k+2, k\geqslant 2\}\cap\{n|n=7k+3, k\geqslant 1\}=\{n|n=21k+17, k\geqslant 0\}$, we have that

$$h(P_2)h(P_6)|h(B_{21(k-1)+10,1,3}). \tag{4.2}$$

From Lemma 3.1.1, it follows that $h(P_2)|h(P_{20})$ and $h(P_6)|h(P_{20})$, together with $(h_1(P_2), h_1(P_6))=1$, which leads to

$$h(P_2)h(P_6)|h(P_{20}). \tag{4.3}$$

From (4.1) to (4.3), we arrive at $h(P_2)h(P_6)|h(B_{21k+10,1,3})$, together with Theorem 3.1.3, we know that the theorem holds. □

Theorem 4.2. *Let graph G such that $G\overset{h}{\sim}B_{n-7,1,3}$, where $n\geqslant 8$. Then*:

(1) *if $n=8$, then $[G]_h=\{B_{1,1,3}, Q_{2,3}\}$,*
(2) *if $n=9$, then $[G]_h=\{B_{2,1,3}, Q_{1,5}, C_7(P_3), C_4(P_6)\}$,*
(3) *if $n=13$, then $[G]_h=\{B_{6,1,3}, C_7\cup B_{1,1,1}, C_7\cup Q_{1,2}, C_7\cup C_4(P_3)\}$,*
(4) *if $n=14$, then $[G]_h=\{B_{7,1,3}, D_6\cup C_7(P_2)\}$,*
(5) *if $n=17$, then $[G]_h=\{B_{10,1,3}, C_6(P_2)\cup P_6\cup K_4^-, B_{2,1,1}\cup P_6\cup K_4^-\}$,*
(6) *if $n\neq 8, 9, 13, 14, 17$, then $[G]_h=\{B_{n-7,1,3}\}$.*

Proof. (1) When $n=8$, let graph G such that $h(G)=h(B_{1,1,3})$. From Lemmas 2.1, 2.2 and 2.6, we obtain that $p(G)=q(G)=8$ and $R_1(G)=-1$. We distinguish the following cases:

Case 1: G is a connected graph.

From $b_3(G)=b_3(B_{1,1,3})$ and (2) of Lemma 2.11, it follows that $G\in\{Q_{2,3}, B_{1,1,3}\}$, where $b_3(G)$ is the fourth coefficient of $h(G)$. By calculation, we have that

$$Q_{2,3}, B_{1,1,3}\in[G]_h.$$

Case 2: G is not a connected graph.

By calculation, we obtain that $h(G)=h(B_{1,1,3})=x^4f_1(x)f_2(x)$, where $f_1(x)=(x+1)$ and $f_2(x)=(x^3+7x^2+12x+3)$. Note that $R_1(f_1(x))=1$ and $b_1(f_1(x))=1$, from (1) of Lemma 2.6, we have that $f_1(x)=h_1(P_2)$ if $f_1(x)$ is a factor of adjoint polynomial of some graph.

Case 2.1: P_2 is not a component of G.

Since G is not connected, then the expression of G is $G=aK_1\cup G_1$, where $a\geqslant 1$ and G_1 is a connected graph. It is not difficult to obtain that $q(G_1)-p(G_1)\geqslant 1$. Noting that $R_1(G_1)=-1$, we have, from Lemma 2.7, that $q(G_1)-p(G_1)\leqslant 1$. Thus $q(G_1)=8=p(G_1)+1$, which leads to $G_1\cong F_7$ by Lemma 2.6. By calculation, we arrive at $h(G)=h(K_1\cup F_7)\neq h(B_{1,1,3})$.

Case 2.2: P_2 is a component of G.

In terms of $h_1(P_2)|h(G)$ and $h_1^2(P_2)\nmid h(G)$, so G only has one component P_2. Let $G=P_2\cup G_1$, where $h(G_1)=x^3(x^3+7x^2+12x+3)$ which results in $R_1(G_1)=-2$ and $q(G_1)=p(G_1)+1$. Thus the following two subcases to be discussed:

Subcase 2.1: G_1 is a connected graph.

From (4) of Lemma 2.6, we arrive at $G_1\in\psi$. From Lemma 2.12, we have that $b_3(G_1)\geqslant 4$, which contradicts to $b_3(G_1)=3$.

Subcase 2.2: G_1 is not a connected graph.

From $h_1(G_1,1)=23$ and Lemma 2.13, we obtain that $h_1(G)$ is a irreducible polynomial over the rational number field, which leads to $G_1\cong aK_1\cup G_2$ and $G=P_2\cup aK_1\cup G_2$, where $a\geqslant 1$ and G_2 is a connected graph. It is not difficult to get that $q(G_2)-p(G_2)\geqslant 2$. From (1) of Lemma 2.7, we have that $q(G_2)-p(G_2)\leqslant 2$. Hence $q(G_2)-p(G_2)=2$, which leads to $G_2\cong K_4$ by (4) of Lemma 2.6. This contradicts to $q(G_2)=7$.

(2) When $n=9$, let graph G such that $h(G)=h(B_{2,1,3})$, which leads to $p(G)=q(G)=9$ and $R_1(G)=-1$. We distinguish the following cases:

Case 1: G is a connected graph.

By $b_3(G)=b_3(B_{2,1,3})$ and (1) of Lemma 2.11, we obtain that $G\in\{C_4(P_6), C_5(P_5), C_6(P_4), C_7(P_3), Q_{1,5}, B_{2,1,3}, B_{3,1,2}\}$. By calculation, we have that

$$C_4(P_6), C_7(P_3), Q_{1,5}, B_{2,1,3}\in[G]_h.$$

Case 2: G is not a connected graph.

By calculation, we have that $h(G)=h(B_{2,1,3})=x^5(x^4+9x^3+26x^2+27x+8)$. According to $h_1(G,1)=71$ and Lemma 2.13, we arrive at $h_1(G)$ is a irreducible polynomial over the rational number field, which leads to $G=aK_1\cup G_1$, where $a\geqslant 1$ and G_1 is a connected graph. It is easy to get that $q(G_1)-p(G_1)\geqslant 1$. From $R_1(G_1)=-1$ and (1) of Lemma 2.7, it follows that $q(G_1)-p(G_1)\leqslant 1$, which leads to $G_1\cong F_8$ by Lemma 2.6. Thus $G=K_1\cup F_8$, which contradicts to $h(G)=h(B_{2,1,3})$.

(3) When $n=10$, let graph G such that $h(G)=h(B_{3,1,3})$, which brings about $p(G)=q(G)=10$ and $R_1(G)=-1$. We distinguish the following cases:

Case 1: G is a connected graph.

By $b_3(G)=b_3(B_{3,1,3})$, we have that $G\in\{C_4(P_7), C_5(P_6), C_6(P_5), C_7(P_4), C_8(P_3), Q_{1,6}, B_{2,1,4}, B_{3,1,3}, B_{4,1,2}\}$. By calculation, we have that $h(G)=h(B_{3,1,3})$ if and only if $G\cong B_{3,1,3}$, which implies that $B_{3,1,3}$ is adjoint uniqueness.

Case 2: G is not a connected graph.

By calculation, we obtain that $h(B_{3,1,3})=x^5f_1(x)f_2(x)$, where $f_1(x)=x^3+5x^2+6x+1$ and $f_2(x)=x^2+5x+3$. Remarking that $R_1(f_1(x))=1$ and $b_1(f_1(x))=5$, from (1) of Lemma 2.6, we arrive at $f_1(x)=h_1(P_6)$ if $f_1(x)$ is a factor of adjoint polynomial of some graph.

Case 2.1: P_6 is not a component of G.

Since G is not connected, then the expression of G is $G=aK_1\cup G_1$, where $a\geqslant 1$ and G_1 is a connected graph. It is not difficult to obtain that $q(G_1)-p(G_1)\geqslant 1$. Noting that $R_1(G_1)=-1$, we have, from Lemma 2.7, that $q(G_1)-p(G_1)\leqslant 1$. Thus $q(G_1)=10=p(G_1)+1$, which leads to $G_1\cong F_9$ by Lemma 2.6. By calculation, we arrive at $h(G)=h(K_1\cup F_9)\neq h(B_{3,1,3})$.

Case 2.2: P_6 is a component of G.

From $h_1(P_6)|h(G)$ and $h_1^2(P_6)\nmid h(G)$, it follows that G only has one component P_6. Let $G=P_6\cup G_1$, where $h(G_1)=x^2(x^2+5x+3)$. The following subcases to be considered:

Subcase 2.1: G_1 is a connected graph.

From $R_1(G_1)=-2$ and $q(G_1)=p(G_1)+1$, we have that $G_1\in\psi$ by (4) of Lemma 2.6. It is the fact that the order of any graph belonging to ψ is not less than 5, which contradicts to $p(G_1)=4$.

Subcase 2.2: G_1 is not a connected graph.

From $h_1(G_1,2)=17$ and Lemma 2.13, we have that $h_1(G_1)$ is a irreducible polynomial over the rational number field, which leads to $G_1=aK_1\cup G_2$ and $G=P_6\cup aK_1\cup G_2$, where $a\geqslant 1$ and G_2 is a connected graph. It is not difficult to get that $q(G_2)-p(G_2)\geqslant 2$. From (1) of Lemma 2.7, we have that $q(G_2)-p(G_2)\leqslant 2$. Hence $q(G_2)-p(G_2)=2$, which leads to $G_2\cong K_4$ by (4) of Lemma 2.6. This contradicts to $q(G_2)=5$.

(4)When $n=11$, Using the similar method as that of (3), we can show that $B_{4,1,3}$ is adjoint uniqueness. The details of the proof are omitted.

(5) When $n\geqslant 12$, let $G=\bigcup_{i=1}^t G_i$. From Lemma 2.1, we have that

$$h(G)=\prod_{i=1}^{t}h(G_i)=h(B_{n-7,1,3}),\tag{4.4}$$

which results in $\beta(G)=\beta(B_{n-7,1,3})\in[-2-\sqrt{5},-4)$ by Lemma 2.10. In terms of $h_1(P_4)=h_1(C_3)$ and Theorem 3.1.2, we know that G contains no C_3 as its component. Let s_i denote the number of components G_i such that $R_1(G_i)=-i$, where $i\geqslant -1$. From Theorem 4.1, Lemmas 2.1 and 2.2, it follows that $0\leqslant s_{-1}\leqslant 2$, $R_1(G)=\sum_{i=1}^t R_1(G_i)=-1$ and $q(G)=p(G)$, which results in

$$\begin{aligned}&-3\leqslant R_1(G_i)\leqslant 1,\\ &s_{-1}=s_1+2s_2+3s_3-1,\\ &\sum_{-3\leqslant R_1(G_i)\leqslant 0}(q(G_i)-p(G_i))=s_{-1}.\end{aligned}\tag{4.5}$$

According to (4.5) and Lemma 2.7, we have that

$$-1+s_3+s_1\leqslant\sum_{R_1(G_i)=-1}(q(G_i)-p(G_i))\leqslant s_1,\tag{4.6}$$

We distinguish the following cases by $0\leqslant s_{-1}\leqslant 2$:

Case 1: $s_{-1}=0$.

It follows, from (4.5) and (4.6), that

$$s_3=0,\quad s_2=0,\quad s_1=1,\quad \text{and}\quad 0\leqslant q(G_1)-p(G_1)\leqslant 1 \text{ with } R_1(G_1)=-1. \tag{4.7}$$

From (4.7), we set

$$G=G_1\cup\left(\bigcup_{i\in A}C_i\right)\cup\left(\bigcup_{j\in B}D_i\right)\cup fD_4\cup aK_1\cup bT_{1,1,1}\cup\left(\bigcup_{T\in\mathscr{T}_0}T_{l_1,l_2,l_3}\right), \tag{4.8}$$

where $\cup_{T\in\mathscr{T}_0}T_{l_1,l_2,l_3}=(\cup_{T\in\mathscr{T}_1}T_{1,1,l_3})\cup(\cup_{T\in\mathscr{T}_2}T_{1,l_2,l_3})\cup(\cup_{T\in\mathscr{T}_3}T_{l_1,l_2,l_3})$, $\mathscr{T}_1=\{T_{1,1,l_3}|l_3\geqslant 2\}$, $\mathscr{T}_2=\{T_{1,l_2,l_3}|l_3\geqslant l_2\geqslant 2\}$, $\mathscr{T}_3=\{T_{l_1,l_2,l_3}|l_3\geqslant l_2\geqslant l_1\geqslant 2\}$, $\mathscr{T}_0=\mathscr{T}_1\cup\mathscr{T}_2\cup\mathscr{T}_3$, the tree T_{l_1,l_2,l_3} is denoted by T for short, $A=\{i|i\geqslant 4\}$ and $B=\{j|j\geqslant 5\}$.

From Theorems 3.1.4, 3.1.5 and 3.1.6, we arrive at

$$R_4(G)=R_4(B_{n-7,1,3})=4=R_4(G_1)+|B|+a+|\mathscr{T}_1|+2|\mathscr{T}_2|+3|\mathscr{T}_3|. \tag{4.9}$$

We distinguish the following cases by $0\leqslant q(G_1)-p(G_1)\leqslant 1$:

Case 1.1: $q(G_1)=p(G_1)+1$.

From Lemmas 2.6 and 2.10, we have $G_1\in\{F_m, K_4^-|m\geqslant 9\}$. Recalling that $q(G)=p(G)$, we obtain the following equality:

$$a+b+|\mathscr{T}_1|+|\mathscr{T}_2|+|\mathscr{T}_3|=1. \tag{4.10}$$

Subcase 1.1.1: $G_1\cong F_m$.

If $m\geqslant 9$, from (3) of Theorem 3.1.6, (4.9) and (4.10), we arrive at $|B|+a+|\mathscr{T}_1|+2|\mathscr{T}_2|+3|\mathscr{T}_3|=1$, which leads to $|B|+a+|\mathscr{T}_1|=1$, $|\mathscr{T}_2|=|\mathscr{T}_3|=0$ and $a+b+|\mathscr{T}_1|=1$. Thus, we have the following three cases to be considered:

If $|B|=1$, then $a=|\mathscr{T}_1|=0$ and $b=1$, which results in

$$G=F_m\cup\left(\bigcup_{i\in A}C_i\right)\cup D_j\cup fD_4\cup T_{1,1,1}.$$

If $a=1$, then $|B|=|\mathscr{T}_1|=b=0$, which leads to

$$G=F_m\cup\left(\bigcup_{i\in A}C_i\right)\cup fD_4\cup K_1.$$

If $|\mathscr{T}_1|=1$, then $|B|=a=b=0$, which brings about

$$G=F_m\cup\left(\bigcup_{i\in A}C_i\right)\cup fD_4\cup T_{1,1,l_3}.$$

As stated above, we always have, from Lemmas 2.9 and 2.10, that $\beta(G)=\beta(F_m)$. From (2) of Theorem 3.2.1 and $\beta(G)=\beta(B_{n-7,1,3})$, it follows that $\beta(F_m)=\beta(B_{n-7,1,3})$ if and only if $m=7$, $n=10$, or $m=9$, $n=13$, or $m=11$, $n=17$. Note that $p(G)=p(B_{n-7,1,3})=n$, so we only have $G=F_{11}\cup C_5\cup K_1$, or $G=F_{11}\cup D_5\cup K_1$, or $G=F_{11}\cup T_{1,1,3}$, which contradicts to $h(G)=h(B_{10,1,3})$ by direct calculation.

Subcase 1.1.2: $G_1\cong K_4^-$.

From (3) of Theorem 3.1.6, (4.9) and (4.10), it follows that $|B|+a+|\mathscr{T}_1|+2|\mathscr{T}_2|+3|\mathscr{T}_3|=2$, which leads to $|\mathscr{T}_3|=0$ and $a+b+|\mathscr{T}_1|+|\mathscr{T}_2|=1$. Hence, the following two subcases to be discussed by the above two equalities:

Subcase 1.1.2.1: $|\mathscr{T}_2|=1$.

Then $|B|=a=|\mathscr{T}_1|=0$ and $b=0$, which leads to

$$G=K_4^-\cup\left(\bigcup_{i\in A}C_i\right)\cup fD_4\cup T_{1,l_2,l_3}. \tag{4.11}$$

If $l_2=2$, $2\leqslant l_3\leqslant 5$, or $l_2=l_3=3$, from Lemmas 2.9 and 2.10,, we have $\beta(G)=-4>\beta(B_{n-7,1,3})$, which contradicts to $h(G)=h(B_{n-7,1,3})$.

If $l_2=2$, $l_3\geqslant 6$, or $l_2=3$, $l_3\geqslant 4$, or $l_3\geqslant l_2\geqslant 4$, by Lemmas 2.9, 2.10 and (3) of Theorem 3.3.2, we obtain $\gamma(G)=4$. From $\gamma(G)=\gamma(B_{n-7,1,3})$ and (2) of Theorem 3.3.2, it follows that $n=17$, which brings about $h(G)=h(B_{10,1,3})$. By direct calculation, we obtain that

$$h_1(B_{10,1,3})=h_1(K_4^-)f_1(x)f_2(x),$$

where $f_1(x)=(x^3+5x^2+6x+1)$ and $f_2(x)=(x^3+7x^2+13x+5)$. In terms of Definition 2.1 and calculation, we arrive at $R_1(f_1)=1$ and $R_1(f_2)=-1$, which contradicts to $R_1(C_i)=R_1(D_4)=R_1(1,l_2,l_3)=0$ in (4.11).

Subcase 1.1.2.2: $|\mathscr{T}_2|=0$.

If $|B|=2$, then $a=|\mathscr{T}_1|=0$ and $b=1$, which leads to

$$G=K_4^-\cup\left(\bigcup_{i\in A}C_i\right)\cup fD_4\cup\left(\bigcup_{j\in B}D_j\right)\cup T(1,1,1).$$

If $|B|=a=1$, then $|\mathscr{T}_1|=b=0$, which results in

$$G=K_4^-\cup\left(\bigcup_{i\in A}C_i\right)\cup fD_4\cup D_j\cup K_1.$$

If $|B|=|\mathscr{T}_1|=1$, then $a=b=0$, which brings about

$$G=K_4^-\cup\left(\bigcup_{i\in A}C_i\right)\cup fD_4\cup D_j\cup T_{1,1,l_3}.$$

As stated above, if $4\leqslant j\leqslant 8$, or $j\geqslant 9$, from Lemmas 2.9,, 2.10 and (4) of Theorem 3.2.1, we have that $\beta(G)=-4>\beta(B_{n-7,1,3})$, or $\beta(G)=\beta(D_j)>\beta(B_{n-7,1,3})$, which contradicts to $h(G)=h(B_{n-7,1,3})$.

Case 1.2: $q(G_1)=p(G_1)$.

Recalling that $q(G)=p(G)$, we arrive at, from (4.8), $a=b=|\mathscr{T}_1|=|\mathscr{T}_2|=|\mathscr{T}_3|=0$, which leads to

$$G=G_1\cup\left(\bigcup_{i\in A}C_i\right)\cup\left(\bigcup_{j\in B}D_i\right)\cup fD_4. \tag{4.12}$$

From (3) of Lemma 2.6 and Lemma 2.10, it follows that

$$G_1\in\{B_{m-t-4,1,t},C_r(P_2),Q_{1,2},C_4(P_3)\}, \tag{4.13}$$

where $m-t-4$, t and r satisfy the conditions of Lemma 2.10.

We distinguish the following cases by (4.13):

Subcase 1.2.1: $G_1\cong C_r(P_2)$.

From Lemma 2.9 and (1) of Lemma 3.2.2, it follows that $\beta(G)=\beta(C_r(P_2))$. Since $\beta(G)=\beta(B_{n-7,1,3})$, we have, from (1) of Theorem 3.2.1, that $\beta(B_{n-7,1,3})=\beta(C_r(P_2))$ if and only if $n=12$, $r=14$, or $n=13$, $r=8$, or $n=14$, $r=7$, or $n=17$, $r=6$. The four subcases to be discussed:

Subcase 1.2.1.1: $n=12$, $r=14$.

In this subcase, it contradicts to $p(G)=p(B_{n-7,1,3})$.

Subcase 1.2.1.2: $n=13$, $r=8$.

From (4.12) and $p(G) = 13$, we only have that $G = C_8(P_2) \cup C_4$ or $G = C_8(P_2) \cup D_4$, which contradicts to $h(G) = h(B_{6,1,3})$.

Subcase 1.2.1.3: $n = 14, r = 7$.

From (4.12) and $p(G)=14$, we only obtain that $G=C_7(P_2)\cup C_6$ or $G=C_7(P_2)\cup D_6$. By Lemma 2.3 and calculation, it follows that $C_7(P_2) \cup D_6 \in [G]_h$.

Subcase 1.2.1.4: $n = 17, r = 6$.

By (4.12) and $p(G) = 17$, we arrive at $G \in \{C_6(P_2) \cup C_{10}, C_6(P_2) \cup D_{10}, C_6(P_2) \cup C_4 \cup C_6, C_6(P_2) \cup C_4 \cup D_6, C_6(P_2) \cup D_4 \cup D_6, C_6(P_2) \cup D_4 \cup C_6\}$, which contradicts to $h(G) = h(B_{10,1,3})$ by calculation.

Subcase 1.2.2: $G_1 \cong Q_{1,2}$ or $G_1 \cong C_4(P_3)$.

From (6) of Theorem 3.2.1 and Lemma 2.9, we have that $\beta(G)=\beta(G_1)$. By (5) of Theorem 3.2.1, $\beta(G)=\beta(B_{n-7,1,3})$ if and only if $n=13$, which leads to $G \in \{Q_{1,2} \cup C_7, Q_{1,2} \cup D_7, C_4(P_3) \cup C_7, C_4(P_3) \cup D_7\}$ by (4.12). By calculation, we have $Q_{1,2} \cup C_7, C_4(P_3) \cup C_7 \in [G]_h$ and $p(G) = 13$.

Subcase 1.2.3: $G_1 \cong B_{m-t-4,1,t}$.

We distinguish the following subcases:

Subcase 1.2.3.1: $t = 1$.

From Lemma 2.9 and (3) of Lemma 3.2.2, it follows that $\beta(G) = \beta(B_{m-5,1,1})$. According to (3) of Theorem 3.2.1, we obtain that $\beta(B_{m-5,1,1}) = \beta(B_{n-7,1,3})$ if and only if $m = 6, n = 13$, or $m = 7, n = 17$, which leads to $G \in \{B_{1,1,1} \cup C_7, B_{1,1,1} \cup D_7\}$, or $G \in \{B_{2,1,1} \cup C_{10}, B_{2,1,1} \cup D_{10}, B_{2,1,1} \cup C_4 \cup C_6, B_{2,1,1} \cup C_4 \cup D_6, B_{2,1,1} \cup D_4 \cup C_6, B_{2,1,1} \cup D_4 \cup D_6, B_{2,1,1} \cup 2C_5, B_{2,1,1} \cup 2D_5\}$ from (4.12). By direct calculation, we only have that $B_{1,1,1} \cup C_7 \in [G]_h$ and $p(G) = 13$.

Subcase 1.2.3.2: $t = 2$.

From (2) of Theorem 3.2.1, (3) of Lemmas 3.2.2 and 2.9, it follows that $\beta(G) = \beta(B_{m-2,1,2}) = \beta(B_{n-7,1,3})$ if and only if $m = 10, n = 13$, or $m = 12, n = 17$, which leads to $G \in \{B_{6,1,2} \cup C_5, B_{6,1,2} \cup D_5\}$ from (4.12). By calculation, we know that this contradicts to $h(G) = h(B_{10,1,3})$.

Subcase 1.2.3.3: If $t \geqslant 4$, from (4) and (7) of Theorem 3.2.1, we arrive at $\beta(G)=\beta(B_{m-t-4,1,t}) < \beta(B_{n-7,1,3})$, which contradicts to $\beta(G) = \beta(B_{n-7,1,3})$.

As analyzed above, we obtain that $t=3$. From (4) of Theorem 3.2.1 and Lemma 2.9, it follows that $\beta(G)=\beta(B_{m-7,1,3})$, together with $\beta(G) = \beta(B_{n-7,1,3})$ and (5) of Lemma 3.2.4, we arrive at $m = n$. Hence $G \cong B_{n-7,1,3}$.

Case 2: $s_{-1} = 1$.

It follows, from (4.5), that $s_1 + 2s_2 + 3s_3 = 2$, which leads to $s_3 = 0$ and $s_1 + 2s_2 = 2$. Hence

$$s_2 = 1, s_1 = 0 \quad \text{or} \quad s_2 = 0, s_1 = 2. \tag{4.14}$$

We distinguish the following cases by (4.14):

Case 2.1: $s_2 = 1, s_1 = 0$.

Without loss of generality, let the component G_1 such that $R_1(G_1) = -2$. From Corollary 2.1, we know that $\beta(G_1) < -2 - \sqrt{5}$, which contradicts to $\beta(B_{n-7,1,3}) \in [-2 - \sqrt{5}, -4)$.

Case 2.2: $s_2 = 0, s_1 = 2$.

Without loss of generality, Let

$$G = G_1 \cup G_2 \cup G_3 \cup \left(\bigcup_{i\in A} C_i\right) \cup \left(\bigcup_{j\in B} D_i\right) \cup f D_4 \cup a K_1 \cup b T_{1,1,1} \cup \left(\bigcup_{T\in\mathscr{T}_0} T_{l_1,l_2,l_3}\right), \tag{4.15}$$

where $\cup_{T\in\mathscr{T}_0} T_{l_1,l_2,l_3} = (\cup_{T\in\mathscr{T}_1} T_{1,1,l_3}) \cup (\cup_{T\in\mathscr{T}_2} T_{1,l_2,l_3}) \cup (\cup_{T\in\mathscr{T}_3} T_{l_1,l_2,l_3})$, $\mathscr{T}_1 = \{T_{1,1,l_3} | l_3 \geqslant 2\}$, $\mathscr{T}_2 = \{T_{1,l_2,l_3} | l_3 \geqslant l_2 \geqslant 2\}$, $\mathscr{T}_3 = \{T_{l_1,l_2,l_3} | l_3 \geqslant l_2 \geqslant l_1 \geqslant 2\}$, $\mathscr{T}_0 = \mathscr{T}_1 \cup \mathscr{T}_2 \cup \mathscr{T}_3$, the tree T_{l_1,l_2,l_3} is denoted by T for short, $G_1 \in \{P_2, P_6\}$, $R_1(G_2) = R_2(G_3) = -1$, $A = \{i | i \geqslant 4\}$ and $B = \{j | j \geqslant 5\}$.

From (4.6), we obtain that

$$1 \leqslant \sum_{i=2}^{3} (q(G_i) - p(G_i)) \leqslant 2. \tag{4.16}$$

We distinguish the following cases by (4.16):

Subcase 2.2.1: $\sum_{i=2}^{3} (q(G_i) - p(G_i)) = 1$.

From Lemmas 2.6 and 2.10, it follows that $G_2 \in \{C_m(P_2), B_{r,1,t}, C_4(P_3), Q_{1,2}\}$ and $G_3 \in \{F_s, K_4^- | s \geqslant 9\}$, where s, r and t satisfy the conditions of Lemma 2.10. Recalling that $q(G) = p(G)$, we have, from (4.15), that

$$a = b = |\mathscr{T}_1| = |\mathscr{T}_2| = |\mathscr{T}_3| = 0,$$

which implies $G = G_1 \cup G_2 \cup G_3 \cup (\cup_{i \in A} C_i) \cup (\cup_{j \in B} D_i) \cup f D_4$ and $R_4(G) = 4 = R_4(G_1) + R_4(G_2) + R_4(G_3) + |B|$. In terms of Theorem 3.1.6, we have that $R_4(G) = 4$ if and only if $G_1 \cong P_6$, $G_3 \cong K_4^-$, $|B| = 0$ and $R_4(G_2) = 3$, which brings about

$$G = P_6 \cup G_2 \cup K_4^- \cup \left(\bigcup_{i \in A} C_i \right) \cup f D_4 \quad \text{and} \quad G_2 \in \{C_s(P_2), B_{r,1,t}\}, \tag{4.17}$$

where r, s and t satisfy the conditions of Lemma 2.10.

Let $p(G_2) = m$, From (4.17), it follows that $m \leqslant n - 10$. We distinguish the following cases by (4.17):

Subcase 2.2.1.1: $G_2 \cong C_s(P_2)$.

From Lemmas 2.9 and 2.10, we obtain that $\beta(G) = \beta(C_s(P_2))$. By (1) of Theorem 3.2.1 and $m \leqslant n - 10$, we arrive at $\beta(C_s(P_2)) = \beta(B_{n-7,1,3})$ if and only if $s = 6$ and $n = 17$. By calculation, we have that

$$P_6 \cup C_6(P_2) \cup K_4^- \in [G]_h \quad \text{and} \quad p(G) = 17.$$

Subcase 2.2.1.2: $G_2 \cong B_{r,1,t}$.

By Lemmas 2.9 and 2.10, we have that $\beta(G) = \beta(B_{r,1,t})$.

If $t = 1$, from (3) of Theorem 3.2.1 and $m \leqslant n - 10$, it follows that $\beta(B_{r,1,1,}) = \beta(B_{n-7,1,3})$ if and only if $r = 2$ and $n = 17$, which leads to

$$G = P_6 \cup B_{2,1,1} \cup K_4^- \in [G]_h \quad \text{and} \quad p(G) = 17.$$

If $t = 2$, from Lemmas 2.9 and 2.10, we have that $\beta(B_{r,1,2}) = \beta(G) = \beta(B_{n-7,1,3})$, which must contradict to $m \leqslant n - 10$ by (2) of Theorem 3.2.1.

If $t = 3$, from Lemmas 2.9 and 2.10, we obtain that $\beta(B_{r,1,3}) = \beta(G) = \beta(B_{n-7,1,3})$, which is a contradiction by $m < n$ and (5) of Lemma 3.2.4.

If $t \geqslant 4$, we can turn to Subcase 1.2.3.3 for the same contradiction.

Subcase 2.2.2: $\sum_{i=2}^{3} (q(G_i) - p(G_i)) = 2$.

From Lemmas 2.6 and 2.10, we have that $G_i \in \{F_m, K_4^- | m \geqslant 9\}$ for $i = 2, 3$.

If $G_2, G_3 \in \{F_m | m \geqslant 9\}$, by (3) of Theorem 3.1.6, it follows that $R_4(G_2) + R_4(G_3) = 6$. From (4.15), we have that $R_4(G) = 4 = R_4(G_1) + R_4(G_2) + R_4(G_3) + |B| + a + |\mathscr{T}_1| + 2|\mathscr{T}_2| + 3|\mathscr{T}_3|$, which leads to $|B| + r + |\mathscr{T}_1| + 2|\mathscr{T}_2| + 3|\mathscr{T}_3| \leqslant -1$. This is a contradiction.

If $G_2 \in \{F_m | m \geqslant 9\}$ and $G_3 \in \{K_4^-\}$, from (3) Theorem 3.1.6 and the expression of $R_4(G)$ as above, we obtain that $R_4(G) = 4$ if and only if $|B| = r = |\mathscr{T}_1| = |\mathscr{T}_2| = |\mathscr{T}_3| = 0$ and $G_1 \cong P_6$, which result in $G = P_6 \cup F_m \cup K_4^- \cup (\cup_{i \in A} C_i) \cup f D_4$. From Lemmas 2.9 and 2.10,, we have that $\beta(F_m) = \beta(G) = \beta(B_{n-7,1,3})$, which contradicts to $m \leqslant n - 10$ by (2) of Theorem 3.2.1.

If $G_2, G_3 \in \{K_4^-\}$, from $h(K_4^-) = x^2(x+1)(x+4)$, it follows that $h_1^2(P_2) | h(G)$, that is $h_1^2(P_2) | h(B_{n-7,1,3})$, which contradicts to Theorem3.1.3.

Case 3: $s_{-1} = 2$.

From (4.5), we arrive at $s_1 + 2s_2 + 3s_3 = 3$, which brings about the following cases:

Case 3.1: $s_3 = 1$ and $s_1 = s_2 = 0$.

Let the component G_1 such that $R_1(G_1) = -3$, which contradicts to $\beta(G) \in [-2 - \sqrt{5}, -4)$ by Corollary 2.1.

Case 3.2: $s_2 = 1$ and $s_1 = s_3 = 0$.

According to the same reason as that of case 3.1, we have the contradiction.

Case 3.3: $s_1 = 3$ and $s_2 = s_3 = 0$.

Without loss of generality, from Theorem 4.1, we set

$$G = P_2 \cup P_6 \cup \left(\bigcup_{k=1}^{3} G_k \right) \cup \left(\bigcup_{i \in A} C_i \right) \cup \left(\bigcup_{j \in B} D_i \right) \cup f D_4 \cup a K_1 \cup b T_{1,1,1} \cup \left(\bigcup_{T \in \mathscr{T}_0} T_{l_1,l_2,l_3} \right), \tag{4.18}$$

where $\cup_{T\in\mathscr{T}_0} T_{l_1,l_2,l_3} = (\cup_{T\in\mathscr{T}_1} T_{1,1,l_3}) \cup (\cup_{T\in\mathscr{T}_2} T_{1,l_2,l_3}) \cup (\cup_{T\in\mathscr{T}_3} T_{l_1,l_2,l_3})$, $\mathscr{T}_1 = \{T_{1,1,l_3} | l_3 \geqslant 2\}$, $\mathscr{T}_2 = \{T_{1,l_2,l_3} | l_3 \geqslant l_2 \geqslant 2\}$, $\mathscr{T}_3 = \{T_{l_1,l_2,l_3} | l_3 \geqslant l_2 \geqslant l_1 \geqslant 2\}$, $\mathscr{T}_0 = \mathscr{T}_1 \cup \mathscr{T}_2 \cup \mathscr{T}_3$, the tree T_{l_1,l_2,l_3} is denoted by T for short, $R_1(G_k) = -1$ for $1 \leqslant k \leqslant 3$, $A = \{i | i \geqslant 4\}$ and $B = \{j | j \geqslant 5\}$.

From (4.6), it follows that

$$2 \leqslant \sum_{k=1}^{3} (q(G_k) - p(G_k)) \leqslant 3. \tag{4.19}$$

We distinguish the following cases by (4.19):

Case 3.3.1: $\sum_{k=1}^{3} (q(G_k) - p(G_k)) = 2$.

From $\beta(G) \in [-2 - \sqrt{5}, -4)$ and Lemmas 2.6 and 2.10, we have that $G_1 \in \{C_s(P_2), B_{r,1,t}, C_4(P_3), Q_{1,2}\}$ and $G_2, G_3 \in \{F_m, K_4^- | m \geqslant 9\}$, where r, s and t satisfy the conditions of Lemma 2.10. Recalling that $q(G) = p(G)$, we have, from (4.18), that

$$a = b = |\mathscr{T}_1| = |\mathscr{T}_2| = |\mathscr{T}_3| = 0,$$

which leads to

$$R_4(G) = 4 = R_4(P_2) + R_4(P_6) + \sum_{k=1}^{3} R_4(G_k) + |B|. \tag{4.20}$$

From Theorem 3.1.6 and (4.20), we obtain that $R_4(G) = 4 \geqslant |B| + 6$, which results in $|B| \leqslant -2$. This is obviously a contradiction.

Case 3.3.2: $\sum_{k=1}^{3} (q(G_k) - p(G_k)) = 3$.

From Lemmas 2.6 and 2.10, we have that $G_k \in \{F_r, K_4^- | r \geqslant 9\}$. By (4.18) and Theorem 3.1.4, we have that

$$R_4(G) = 4 = R_4(P_2) + R_4(P_6) + \sum_{k=1}^{3} R_4(G_k) + a + |B| + |\mathscr{T}_1| + 2|\mathscr{T}_2| + 3|\mathscr{T}_3|. \tag{4.21}$$

In terms of Theorem 3.1.6, we arrive at $R_4(G) = 4 \geqslant 5 + a + |B| + |\mathscr{T}_1| + 2|\mathscr{T}_2| + 3|\mathscr{T}_3|$, which leads to $a + |B| + |\mathscr{T}_1| + 2|\mathscr{T}_2| + 3|\mathscr{T}_3| \leqslant -1$. This is also a contradiction.

This completes the proof of the theorem. □

Corollary 4.1. *For $n \geqslant 8$, graph $B_{n-7,1,3}$ is adjoint uniqueness if and only if $n \neq 8, 9, 13, 14, 17$.*

Corollary 4.2. *For $n \geqslant 8$, the chromatic equivalence class of $\overline{B_{n-7,1,3}}$ only contains the complements of graphs that are in Theorem 4.2.*

Corollary 4.3. *For $n \geqslant 8$, graph $\overline{B_{n-7,1,3}}$ is chromatic uniqueness if and only if $n \neq 8, 9, 13, 14, 17$.*

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